ON THE KERNEL CENTER OF A CONVEX BODY

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Abstract. It is well known that the set of centers of minimal balls containing a convex body is a singleton, but the set of incenters of that body, i.e., its kernel, need not be a singleton. On the other hand, the kernel cannot have the same dimension as the body itself. By *iterating* the construction of the kernel we define a new selector, the *kernel center*, which selects a point from the kernel of a given convex body. Evidently, this selector is constant when restricted to the family of parallel bodies of a fixed convex body. We prove that it is directly additive but not additive, and we study further properties of this selector.

1. Introduction

A selector for a family $\mathcal{X}$ of subsets of a metric space is a function on $\mathcal{X}$ which selects a point from every member of this family. For $n \geq 1$, we deal with the family $\mathcal{K}_0^n$ of convex bodies in R^n, that is, of compact convex subsets of R^n with nonempty interiors.

Selectors for $\mathcal{K}_0^n$ have been studied by many authors (see [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) .

The present paper concerns kernels of convex bodies in R^n. Using this notion, we define a new selector, the *kernel center map*, whose image belongs to the kernel of the convex body. This selector is constant when restricted to the family of inner parallel bodies of a given convex body (see next Section for definitions). For a convex body A this new selector is a relative of the Chebyshev center, $\check{c}$ (see [7]), which is the center of the unique ball in R^n with minimal radius, containing A. Since, generally, for a convex body $A \subset \mathrm{R}^n$ a ball with maximal radius contained in A is not unique, there is no analogue to the Chebyshev center with balls containing A replaced by balls contained in A. The kernel center map selects the incenter of one of the largest balls contained in a convex body. Of course, it coincides with its incenter if the convex body has a unique largest ball contained in it.

We warn the reader that the notion of *kernel* is commonly used for star bodies in a quite different meaning (see, for instance, [4],[7] and [10]). For convex bodies these two notions of kernel differ essentially.

In Section 3 we study the Minkowski additivity and direct additivity of kernels of convex bodies. In Section 4 we focus on the kernel center map and prove that it is neither continuous with respect to the Hausdorff metric nor Minkowski additive. In turn, we prove that the kernel center map is directly additive (Theorem 4.1).

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In the last section we compare the kernel center with some well known selectors: the centroid, the Steiner point, the center of the minimal ring, the Chebyshev center, and the pseudocenter, proving that the kernel center coincides, in general, with none of them.

2. Preliminaries.

We use the following terminology and notation. Let A be a nonempty subset of $\mathrm{R^n}$. As usual, aff$A$, lin$A$, int$A$, relint$A$, cl A, conv$A$ are, respectively, the affine hull of A, the linear hull of A, the interior of A, the relative interior of A, the closure of A, and the convex hull of A. The (closed) unit ball of $\mathrm{R^n}$ is denoted by B^n, and the Lebesgue measure by λ_n.

Let $\mathcal{K}^n$ be the family of nonempty compact convex subsets of $\mathrm{R^n}$ and $\mathcal{K}_0^n := \{A \in \mathcal{K}^n \mid \mathrm{int}A \neq \emptyset\}$. Let E be an affine subspace of $\mathrm{R^n}$. The map $\pi_E : \mathrm{R^n} \to E$ is the orthogonal projection of $\mathrm{R^n}$ onto E and $\mathcal{K}_0(E) = \{A \in \mathcal{K}^n \mid \mathrm{relint}_E A \neq \emptyset\}$.

For any nonempty $A_1, A_2 \subset \mathrm{R^n}$, the Minkowski sum, $A_1 + A_2$, and the Minkowski difference, $A_1 \dot{-} A_2$, are defined by

$$A_1 + A_2 := \{a_1 + a_2 \mid a_j \in A_j \text{ for } j = 1, 2\} \quad \text{and}$$
$$A_1 \dot{-} A_2 := \{x \in \mathrm{R^n} \mid x + A_2 \subset A_1\}.$$

It follows from the definition that if $A_1, A_2 \in \mathcal{K}_0^n$, then $A_1 + A_2 \in \mathcal{K}_0^n$, and if, moreover, $A_2 \subset A_1$, then $A_1 \dot{-} A_2 \in \mathcal{K}^n$.

For a convex body $A \in \mathcal{K}_0^n$, the *inradius* $\mathrm{r}(A)$ is defined by

$$\mathrm{r}(A) := \sup\{r \mid \exists x \in \mathrm{R^n} \ : \ x + r\, B^n \subset A\}.$$

The *kernel* of A, $\ker(A) = A \dot{-} \mathrm{r}(A) B^n$, is the set of incenters of A. The dimension of $\ker(A)$ is strictly less than n (see [2, p. 59]). The *inner parallel body of A at distance* $0 < \lambda < \mathrm{r}$ is the set $A \dot{-} \lambda B^n$.

Let $E_1, E_2 \subset \mathrm{R^n}$ be two orthogonal affine subspaces of $\mathrm{R^n}$ with $\mathrm{R^n} = E_1 + E_2$. Then $\mathrm{R^n}$ is the direct sum of E_1 and E_2, in symbols $E_1 \oplus E_2$, and for any $A_j \in \mathcal{K}_0(E_j)$, $j = 1, 2$, the direct sum of A_1, A_2 is $A_1 \oplus A_2 := A_1 + A_2 \in \mathcal{K}_0(E)$.

For $A_1, A_2 \in \mathcal{K}^n$, the distance from A_1 to A_2 is given by the Hausdorff metric ρ_H.

Let $k \geq 2$. For affinely independent points $a_1, ..., a_k \in \mathrm{R^n}$, let $\Delta(a_1, \ldots, a_k)$ be the simplex with vertices $a_1, \ldots, a_k$. Finally, $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathrm{R^n}$.

3. The kernel of a convex body.

The following result is evident.

Proposition 3.1. *The function* $\ker : \mathcal{K}_0^n \to \mathcal{K}^n$ *is equivariant under isometries of* $\mathrm{R^n}$.

Remark 3.1. *It is easy to see that the function* ker *is not equivariant under affine maps. It suffices to consider a cube and an orthogonal box with two edges of different lengths.*

It is natural to ask whether the map ker is Minkowski additive. The following example shows that the answer is negative.

Example 3.1.

Consider the following orthogonal boxes $A_1, A_2 \in \mathcal{K}_0^n$*,* $A_1 = \Delta(-e_1, e_1) + \sum_{i=2}^n \Delta(-2e_i, 2e_i)$ *and* $A_2 = \Delta(-2e_1, 2e_1) + \sum_{i=2}^n \Delta(-e_i, e_i)$*. It is easy to check that* $\ker(A_1) = \sum_{i=2}^n \Delta(-e_i, e_i)$ *and* $\ker(A_2) = \Delta(-e_1, e_1)$ *while* $\ker(A_1 + A_2) = \{0\}$*.*

If the convex bodies lie in orthogonal affine flats, i.e., if we are dealing with direct sums, we can say more. For $A \in \mathcal{K}^n$ and $E = \operatorname{aff} A$, r_E and $\ker_E(A)$ denote, respectively, the inradius and the kernel of A in $\mathcal{K}_0(E)$.

Theorem 3.1. *Let* E_1, E_2 *be orthogonal flats with* $\mathrm{R^n} = E_1 \oplus E_2$*. Let* $A_j \in \mathcal{K}_0(E_j)$ *for* $j = 1, 2$*. Then*

i) $\mathrm{r}(A_1 \oplus A_2) = \min_{j=1,2} \mathrm{r}_{E_j}(A_j)$,
ii) $\ker_{E_1}(A_1) + \ker_{E_2}(A_2) \subset \ker_E(A_1 \oplus A_2)$,
iii) $\ker_{E_1}(A_1) + \ker_{E_2}(A_2) = \ker_E(A_1 \oplus A_2)$ *if and only if* $\mathrm{r}_{E_1}(A_1) = \mathrm{r}_{E_2}(A_2)$*.*

Proof. For $j = 1, 2$, let $B_j = \pi_{E_j}(B^n)$. It is easy to see that $B^n \subset B_1 \oplus B_2$.

Let $\mathrm{r}_j := \mathrm{r}_{E_j}(A_j)$ for $j = 1, 2$ and $\mathrm{r} = \mathrm{r}(A_1 \oplus A_2)$. We may assume that $\mathrm{r}_1 \le \mathrm{r}_2$.

i) Let $x \in \ker(A_1 \oplus A_2)$; then $x + \mathrm{r}B^n \subset A_1 \oplus A_2$. Projecting onto E_j, $j \in \{1, 2\}$, we obtain

$$\pi_{E_j}(x) + \mathrm{r}B_j \subset \pi_{E_j}(A_1 \oplus A_2) = A_j$$

and thus $\mathrm{r} \le \mathrm{r}_j$ for $j \in \{1, 2\}$. On the other hand, since $\mathrm{r}_1 \le \mathrm{r}_2$, it follows that $x_j + \mathrm{r}_1 B_j \subset A_j$ for $x_j \in \ker_{E_j}(A_j)$, $j = 1, 2$.

Hence,

$$(x_1 + x_2) + \mathrm{r}_1 B^n \subset x_1 + x_2 + \mathrm{r}_1(B_1 \oplus B_2) \subset A_1 \oplus A_2, \tag{3.1}$$

which proves that $\mathrm{r}_1 \le \mathrm{r}$.

ii) From (3.1) it follows that if $x_j \in \ker_{E_j}(A_j)$, $j = 1, 2$, then $x_1 + x_2 \in \ker(A_1 \oplus A_2)$.

iii) Assume first that $\mathrm{r} := \mathrm{r}_1 = \mathrm{r}_2$. In view of ii), it suffices to prove that $\ker(A_1 \oplus A_2) \subset \ker_{E_1}(A_1) + \ker_{E_2}(A_2)$.

Since $\ker_E(A_1 \oplus A_2) + \mathrm{r}B^n \subset A_1 \oplus A_2$, projecting onto E_j for $j \in \{1, 2\}$, we obtain

$$\pi_{E_j}(\ker_E(A_1 \oplus A_2)) + \mathrm{r}_0 B_j \subset A_j.$$

Hence, $\pi_{E_j}(\ker_E(A_1 \oplus A_2)) \subset \ker_{E_j}(A_j)$ and

$$\pi_{E_1}(\ker_E(A_1 \oplus A_2)) + \pi_{E_2}(\ker_E(A_1 \oplus A_2)) = \ker(A_1 \oplus A_2) \subset \ker_{E_1}(A_1) + \ker_{E_2}(A_2).$$

To prove the converse implication, assume that $\ker_{E_1}(A_1) \oplus \ker_{E_2}(A_2) = \ker(A_1 \oplus A_2)$. From *i)* we know that $\mathrm{r}(A_1 \oplus A_2) = \mathrm{r}_1$. Projecting onto E_j, we obtain $\pi_{E_j}(\ker(A_1 \oplus A_2)) = \ker_{E_j} A_j$ for $j = 1, 2$. If $\mathrm{r}_1 \le \mathrm{r}_2$, then

$$\ker_{E_1}(A_1) + \ker_{E_2}(A_2) + \mathrm{r}_1 B^n + (\mathrm{r}_2 - \mathrm{r}_1)B^n \subset \ker_{E_1}(A_1) + \mathrm{r}_1(B_1 + B_2) + (\mathrm{r}_2 - \mathrm{r}_1)B_2 \subset A_1 \oplus A_2.$$

Thus we get

$$\ker(A_1 \oplus A_2) + \mathrm{r}_1 B^n + (\mathrm{r}_2 - \mathrm{r}_1)B_2 = \ker_{E_1}(A_1) + \ker_{E_2}(A_2) + \mathrm{r}_1 B^n + (\mathrm{r}_2 - \mathrm{r}_1)B_2 \subset A_1 \oplus A_2,$$

which implies $r_1 = r_2$. This completes the proof.

□

Example 3.1 shows that, under the conditions of the previous theorem, if $\mathrm{r}_{E_1}(A_1) \neq \mathrm{r}_{E_2}(A_2)$, then not only the inclusion $\ker_{E_1}(A_1) + \ker_{E_2}(A_2) \subset \ker_E(A_1 \oplus A_2)$ must be strict, but also the inequality

$$\dim \ker_{E_1}(A_1) + \dim \ker_{E_2}(A_2) < \dim \ker(A_1 + A_2)$$

may be strict.

We finish the section with the following remark which is a direct consequence of the definition of inner parallel bodies of $A \in \mathcal{K}_0^n$.

Remark 3.2. *Let* $\{A_\lambda \mid -\mathrm{r}(A) \leq \lambda \leq 0\}$ *be the family of inner parallel bodies of* $A \in \mathcal{K}^n$. *Then* $\ker(A_\lambda) = \ker(A)$ *for all* $\lambda \in [-\mathrm{r}, 0]$.

4. The kernel center of a convex body.

To any convex body $A \in \mathcal{K}_0^n$ we assign two finite sequences, $(\ker^{(i)}(A))_{i \geq 0}$ and $(E_i(A))_{i \geq 0}$, defined as follows:

$$\ker^{(0)}(A) := \ker(A) \quad \text{and} \quad E_0(A) := \operatorname{aff} \ker(A); \tag{4.1}$$

if $i \geq 1$ and $\dim \ker^{(i-1)}(A) > 0$, then

$$\ker^{(i)}(A) := \ker_{E_{i-1}(A)}(\ker^{(i-1)}(A)) \quad \text{and} \quad E_i(A) := \operatorname{aff} \ker^{(i)}(A). \tag{4.2}$$

Let $m(A) := \min\{i \geq 0 \mid \dim \ker^{(i)}(A) = 0\}$. Evidently, the sequences $(E_i(A))_{i \geq 0}$ and $\left(\ker^{(i)}_{E_i}(A)\right)_{i \geq 0}$ are descending.

Moreover, if $\dim \ker^{(i)}(A) > 0$, then $\dim \ker^{(i+1)}(A) < \dim \ker^{(i)}(A)$ (cf. [2, p. 59]).

Then, $\kappa(A)$ is defined as the unique point of $\ker^{m(A)}(A)$, or equivalently:

$$\{\kappa(A)\} = \bigcap_{i=0}^{m(A)} \ker^{(i)}(A). \tag{4.3}$$

Of course $m(A) \in \{0, \ldots, n\}$, and it provides the number of steps needed to reach $\kappa(A)$ when passing from $\ker(A)$ to the subsequent kernels. It is clear that $\kappa(A) \in A$, hence, $\kappa : \mathcal{K}_0^n \to \mathrm{R^n}$ is a selector for $\mathcal{K}_0^n$.

Remark 4.1. *Let us notice that for any affine flat E in $\mathrm{R^n}$, with $\dim E = k \leq n-1$, the functions* $\mathrm{r} : \mathcal{K}_0(E) \to \mathrm{R^n}$, $\ker : \mathcal{K}_0(E) \to \mathcal{K}^n$ *and* $\kappa : \mathcal{K}_0(E) \to \mathrm{R^n}$ *are well defined by identifying E and R^k.*

Since the selector κ is defined by means of the "subsequent kernels" of A, it is natural to ask whether κ and ker behave in a similar way. To answer this, we deal next with some properties of κ. The following statement follows from Lemma 3.1.

Proposition 4.1. *For every isometry* $f : \mathrm{R^n} \to \mathrm{R^n}$ *and every* $A \in \mathcal{K}_0^n$,

$$f(\kappa(A)) = \kappa(f(A)).$$

As it happens with the kernel, the selector κ is not equivariant under affine maps. In fact, it is not affine equivariant when restricted to the family of simplices. For this purpose, we notice that the incenter of a simplex T coincides with $\kappa(T)$. In [3, Theorem 2.1] it is proven that the incenter of a simplex coincides with its centroid if and only if all the facets of the simplex have the same area. The centroid of a convex body is equivariant under affine transformations (see [7, Theorem 12.3.8]). Thus, it is enough to consider any simplex all whose facets do not have the same area, which of course is an affine image of a regular simplex.

Next we prove that κ is not continuous with respect to the Hausdorff metric in $\mathrm{R^n}$ for any $n \geq 2$.

Proposition 4.2. *The selector $\kappa : \mathcal{K}_0^n \to \mathrm{R^n}$ is not continuous with respect to ρ_H.*

Proof. For every natural k, let

$$A_k := \mathrm{conv}(B^n \cup (2e_1 + \frac{k}{1+k} B^n))$$

and $A := \mathrm{conv}(B^n \cup (2e_1 + B^n))$. Evidently $A = \lim_{\rho_H} A_k$ while $e_1 = \kappa(A) \neq \kappa(A_k) = 0$ for any k. □

Let us prove the following statement concerning the kernel center map (cf. Example 3.1).

Proposition 4.3. *The selector $\kappa : \mathcal{K}_0^n \to \mathrm{R^n}$ is not Minkowski additive for any $n \geq 2$.*

Proof. Let A_1 be the orthogonal box $A_1 = \left(\sum_{i=1}^{n-1} \Delta(-\rho\, e_i, \rho\, e_i)\right) \oplus \Delta(-e_n, e_n)$, for $2 < \rho \in \mathbb{R}$ and $A_2 := \mathrm{conv}(B^n \cup \{2\rho\, e_n\})$. It is easy to check that $\ker(A_1) = \sum_{i=1}^{n-1} \Delta(-\rho\, e_i, \rho\, e_i) \subset \mathrm{lin}\{e_1, \dots, e_{n-1}\}$, whence $\kappa(A_1) = 0$. On the other hand, A_2 has only one largest ball centered at the origin. Thus, $\kappa(A_2) = 0$.

It suffices to prove that $0 \notin \ker(A_1 + A_2)$. Let us consider the Minkowski sum $A_1 + A_2 = \left(\sum_{i=1}^{n-1} \Delta(-\rho\, e_i, \rho\, e_i)\right) + \Delta(-e_n, e_n) + \mathrm{conv}(B^n \cup \{2\rho\, e_n\})$. Since $\Delta(-e_n, 2\rho e_n) \subset A_2$, it follows that

$$\begin{aligned} B := &\left(\sum_{i=1}^{n-1} \Delta(-\rho\, e_i, \rho\, e_i)\right) + \Delta(-e_n, e_n) + \Delta(-e_n, 2\rho\, e_n) = \\ &\left(\sum_{i=1}^{n-1} \Delta(-\rho\, e_i, \rho\, e_i)\right) + \Delta(-2e_n, (2\rho+1)e_n) \subset A_1 + A_2. \end{aligned}$$

On the other hand, $\mathrm{r}(B) \geq \rho$. Hence, there exists a ball of radius at least ρ in $A_1 + A_2$ while the largest ball centered at the origin and contained in $A_1 + A_2$ has radius 2. Thus $0 \notin \ker(A_1 + A_2)$, whence $\kappa(A_1 + A_2) \neq 0$. □

The selector κ exhibits a nice behavior when dealing with direct sum.

Theorem 4.1. *Let E_1, E_2 be orthogonal affine flats with $\mathrm{R^n} = E_1 \oplus E_2$. Let $A_1 \in \mathcal{K}_0^n(E_1)$ and $A_2 \in \mathcal{K}_0^n(E_2)$. Then*

$$\kappa(A_1 \oplus A_2) = \kappa(A_1) + \kappa(A_2).$$

Proof. Let $\mathrm{r}_1 = \mathrm{r}_{E_1}(A_1) \leq \mathrm{r}_{E_2}(A_2) = \mathrm{r}_2$ and $\dim E_1 = k$. By Theorem 3.1,

$$\mathrm{r}(A_1 \oplus A_2) = \min\{\mathrm{r}_1, \mathrm{r}_2\} = \mathrm{r}_1.$$

Let us prove that

$$\ker(A_1 \oplus A_2) = \ker_{E_1}(A_1) + (A_2)_{-\mathrm{r}_1}. \tag{4.4}$$

Indeed

$$\ker_{E_1}(A_1) + (A_2)_{-\mathrm{r}_1} + \mathrm{r}_1 B^n \subset A_1 \oplus A_2;$$

thus,

$$\ker_{E_1}(A_1) + (A_2)_{-\mathrm{r}_1} \subset \ker(A_1 \oplus A_2).$$

On the other hand,

$$\ker(A_1 \oplus A_2) + \mathrm{r}_1 B^n \subset A_1 \oplus A_2.$$

Projecting onto E_j, $j = 1, 2$, we obtain $\pi_{E_1}(\ker(A_1 \oplus A_2)) \subset \ker_{E_1}(A_1)$ and $\pi_{E_2}(\ker(A_2 \oplus A_2)) \subset (A_2)_{-\mathrm{r}_1}$. Hence

$$\ker(A_1 \oplus A_2) = \ker_{E_1}(A_1) + (A_2)_{-\mathrm{r}_1},$$

which proves (4.4).

It is clear that dim $\ker_{E_1}(A_1) < k$. Since in each step the dimension of one of the two summands decreases, in order to get $\kappa(A_1 \oplus A_2)$ we need to iterate this process a finite number of steps. After i iterations we will have, for $l, m \in \{1, 2\}$, $l \neq m$, and $j \in \{1, \ldots, i-1\}$, one of the following Minkowski sums: $(A_l)_\mu + \ker^{(i-1)}_{E_m}(A_m)$ for some $-\mathrm{r}_{E_l}(A_l) < \mu < 0$ or $\ker^{j}_{E_l}(A_l) + \ker^{i-1-j}_{E_m}(A_m)$. By Remark 3.2, $\ker_{E_l}((A_l)_\mu) = \ker(A_l)$. Thus, after at most $m_{E_1}(A_1) + m_{E_2}(A_2) + 1$ steps we obtain $\kappa(A_1 \oplus A_2) = \kappa(A_1) + \kappa(A_2)$. □

5. Final remarks

We compare the kernel center map with some well known selectors.

Proposition 5.1. *The kernel center map κ is different from the Steiner point map s, the Chebyshev center $\check{c}$, the centroid c_0, the center of the minimal ring c, and the pseudocenter p.*

Proof.

(1) Let s be the Steiner point map, that is

$$s(A) := \frac{1}{\lambda_n(B^n)} \int_{S^{n-1}} u h_A(u)\, d\sigma(u),$$

where h_A is the support function of the convex body A. Since the Steiner point map s is continuous with respect to ρ_H and Minkowski additive (see e.g. [14]), in view of Propositions 4.2 and 4.3, $\kappa \neq s$.

(2) Let $\check{c}(A)$ be the Chebyshev center of A, i.e., the center of the unique ball with minimal radius containing A. Let A be the cone over the $(n-1)$ - dimensional ball $B^n \cap (\mathrm{line}_n)^\perp$, with vertex e_n. Then $\check{c}(A) = 0 \in \mathrm{bd}A$, while $\kappa(A) \in \mathrm{int}\, A$. Thus $\kappa \neq \check{c}$.

(3) Let $c_0(A)$ be the centroid of A. By [3, Theorem 3.2] the centroid and the incenter of a simplex coincide if and only if all the facets of the simplex have the same area. Hence, $\kappa \neq c_0$.

(4) Let $c(A)$ be center of the minimal ring containing A, that is, the minimizer of $R_A(x) - r_A(x)$, where $R_A(x)$ is the minimal radius of a ball with center x containing A and $r_A(x)$ is the maximal radius of a ball with center x contained in A (cf. [1], [7]). Since the selector c is continuous with respect to the Hausdorff metric ([7, Theorem 12.5.8]), from Proposition 4.2 it follows that $\kappa \neq c$.

(5) Let $p(A)$ be the pseudocenter of A, i.e., the symmetry center of the centrally symmetric convex body with maximal volume contained in A. The selector p is equivariant under affine maps (see [7, Th. 12.6.3]), while the kernel center map is not. Thus $\kappa \neq p$.

□

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